Mid-Semester Exam | ISI Bangalore | February 2020

Algebra IV | 100 Points

**Notes.**

(a) Justify all your steps. You may use any result proved in class unless you have been asked to prove the same.

(b) $\mathbb{N}$ = natural numbers, $\mathbb{Z}$ = integers, $\mathbb{Q}$ = rational numbers, $\mathbb{R}$ = real numbers, $\mathbb{C}$ = complex numbers.

(c) $\mathbb{F}_{p^n}$ denotes a finite field having $p^n$ elements.

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1. [15 points] Let $K = \mathbb{Q}(\alpha)$ where $\alpha$ is a root of $x^3 - 2x - 1$. Determine the irreducible polynomial for $\gamma = 1 + \alpha^2$ over $\mathbb{Q}$.

2. [15 points] Prove that the regular pentagon can be constructed by ruler and compass.

3. [20 points] In each of the following cases, for each integer $n > 0$, determine the number of irreducible factors of degree $n$ for the polynomial $x^{64} - x$ over the given field.

   (a) $\mathbb{F}_2$ (b) $\mathbb{F}_4$ (c) $\mathbb{F}_8$ (d) $\mathbb{F}_{16}$

4. [20 points] Suppose $k \to F$ and $k \to L$ are extensions of fields with $F/k$ algebraic.

   (i) If $L$ is algebraically closed, prove that there is a $k$-linear homomorphism $F \to L$.

   (ii) Is the homomorphism in (i) necessarily unique ? Prove or disprove.

5. [15 points] Let $k$ be a field and let $X_1, \dots, X_m, Y_1, \dots, Y_n$ be independent variables. Let $L = k(\{X_i, Y_j\})$ be the field generated by these variables. Set $\alpha_{ij} := X_i Y_j$. Determine a transcendence basis for the subfield $F = k(\{\alpha_{ij}\})$ over $k$.

6. [15 points] Express $x^3y^3 + x^3z^3 + y^3z^3$ as a polynomial expression in the elementary symmetric polynomials in $x, y, z$.